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## REVIVALS OF RYDBERG WAVE PACKETS <sup>1</sup>

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We examine the revival structure of Rydberg wave packets. These wave packets exhibit initial classical periodic motion followed by a sequence of collapse, fractional/full revivals, and fractional/full superrevivals. The effects of quantum defects on wave packets in alkali-metal atoms and a squeezed-state description of the initial wave packets are also described. We then examine the revival structure of Rydberg wave packets in the presence of an external electric field, i.e., the revival structure of Stark wave packets. These wave packets have energies that depend on two quantum numbers and exhibit new types of interference behavior.

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<sup>1</sup>Talk presented by R.B. at the VIII International Conference on Symmetry Methods in Physics, Dubna, Russia, July, 1997

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## 1. Introduction

A Rydberg wave packet is formed when a short laser pulse excites a coherent superposition of energy states [1, 2]. The initial motion of these localized wave packets is periodic with a periodicity  $T_{\text{cl}}$  equal to the classical period of a particle in a keplerian orbit. However, after several Kepler cycles, the wave packet collapses and a sequence of fractional and full revivals commences. The fractional revivals occur prior to the full revival. They consist of distinct subsidiary waves moving with a period that is a fraction of  $T_{\text{cl}}$  [1, 2, 3]. These culminate with the formation of a full revival at time  $t_{\text{rev}}$  when the wave packet recombines into nearly its original shape. The fractional revivals have been observed in time-delayed photoionization and phase modulation experiments [4].

We begin in the next section by examining the revival structure and evolution of hydrogenic Rydberg wave packets for times much greater than the revival time [5, 6]. We show how a quantum-defect theory based on quantum-mechanical supersymmetry [7] may be used to model wave packets in alkali-metal atoms, and we study the effects of quantum defects on the revivals [5]. We also describe how this analysis of revivals can be applied to other quantum systems [8, 9]. In Sec. 3, we

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show that the motion of these wave packets has features characteristic of squeezed states, and we outline an approach for a squeezed-state description [10, 11]. Sec. 4 examines the revival structure of Rydberg wave packets moving in the presence of an external electric field [12]. These wave packets are referred to as Stark wave packets. To lowest order in a perturbative treatment, Stark wave packets have energies that depend on two quantum numbers. Their evolution is therefore governed by additional periodicities and revival time scales [13]. We prove that under certain conditions Stark wave packets can exhibit full and fractional revivals. We also show that the revivals of Stark wave packets have unique features arising from the fact that the superposition of states can be separated into sums over even and odd values of the principal quantum number  $n$ . The even and odd states evolve differently, which results in new types of interference behavior.

## 2. Superrevivals of Rydberg Wave Packets

To date, experiments that have detected the classical oscillation of a wave packet in a Coulomb potential have been performed using purely radial wave packets, which consist of a superposition of  $n$  states with  $l$  fixed, typically, to a p state. A wave packet of this type is formed when a single short laser pulse excites an atom from its ground state to a superposition of excited states. Radial wave packets follow the initial classical motion, but are localized only in the radial coordinate.

The wave function for a hydrogenic radial wave packet may be expanded in terms of energy eigenstates as

$$\Psi(\vec{r}, t) = \sum_{n=1}^{\infty} c_n \varphi_n(\vec{r}) \exp[-iE_n t] \quad . \quad (1)$$

Here,  $E_n = -1/2n^2$  is the energy in atomic units, and  $\varphi_n(\vec{r})$  represents a radial wave packet,  $\varphi_n(\vec{r}) = \psi_{n,1,0}(\vec{r})$ , where  $\psi_{nlm}(\vec{r})$  is a hydrogen eigenstate of energy and angular momentum. The laser is tuned to excite coherently a superposition of states centered on a central value  $\bar{n}$  of the principal quantum number. The square of the weighting coefficients  $c_n$  is therefore centered on  $\bar{n}$  with a width characterizing the energy spread.

Expanding the energy in a Taylor series around the centrally excited value  $\bar{n}$ , we find that the derivative terms define distinct time scales that depend on  $\bar{n}$ :

$$T_{\text{cl}} = \frac{2\pi}{E'_{\bar{n}}} = 2\pi\bar{n}^3 \quad , \quad (2)$$

$$t_{\text{rev}} = \frac{-2\pi}{\frac{1}{2}E''_{\bar{n}}} = \frac{2\bar{n}}{3}T_{\text{cl}} \quad , \quad (3)$$

$$t_{\text{sr}} = \frac{2\pi}{\frac{1}{6}E'''_{\bar{n}}} = \frac{3\bar{n}}{4}t_{\text{rev}} \quad . \quad (4)$$

Keeping terms through third order, and defining the index  $k = n - \bar{n}$ , we may write the wave function as

$$\Psi(\vec{r}, t) = \sum_{k=-\infty}^{\infty} c_k \varphi_k(\vec{r}) \exp \left[ -2\pi i \left( \frac{kt}{T_{\text{cl}}} - \frac{k^2 t}{t_{\text{rev}}} + \frac{k^3 t}{t_{\text{sr}}} \right) \right] . \quad (5)$$

The time evolution of the wave packet is controlled by the interplay between the three time-dependent terms in the phase. For small times,  $t \lesssim T_{\text{cl}}$ , only the first term in the phase matters and  $\Psi(\vec{r}, t)$  is approximately periodic with period  $T_{\text{cl}}$ . For times  $t \lesssim t_{\text{rev}}$ , the interference of the first two terms causes the formation of fractional revivals and full revivals [3]. These occur at times equal to fractions of  $t_{\text{rev}}$ . The motion of the wave packet at these times is periodic with periods equal to fractions of the classical orbital period  $T_{\text{cl}}$ . The fractional periodicities are evident in plots of the absolute square of the autocorrelation function  $A(t) = \langle \Psi(\vec{r}, 0) | \Psi(\vec{r}, t) \rangle$  as a function of  $t$  [14, 15].

To consider times  $t \lesssim t_{\text{sr}}$ , we must examine the interference between all three terms in the phase of  $\Psi(\vec{r}, t)$ . At certain times  $t_{\text{frac}}$  it is possible to expand the wave function  $\Psi(\vec{r}, t)$  of Eq. (5) as a series of subsidiary wave functions. The idea is to express  $\Psi(\vec{r}, t)$  as a sum of wave functions  $\psi_{\text{cl}}$  having the same periodicities and shape as that of the initial wave function  $\Psi(\vec{r}, 0)$ . We find that at times  $t_{\text{frac}} \approx \frac{1}{q} t_{\text{sr}}$ , where  $q$  must be an integer multiple of 3, the wave packet can be written as a sum of macroscopically distinct wave packets. Furthermore, at these times  $t_{\text{frac}}$ , we also find that the motion of the wave packet is periodic with a period  $T_{\text{frac}} \approx \frac{3}{q} t_{\text{rev}}$ . Note that these periodicities are on a much greater time scale than those of the fractional revivals, and thus a new level of revivals commences for  $t > t_{\text{rev}}$ . We also find that at the time  $t_{\text{frac}} \approx \frac{1}{6} t_{\text{sr}}$ , a single wave packet forms that resembles the initial wave packet more closely than the full revival does at time  $t_{\text{rev}}$ , i.e., a superrevival occurs. In Refs. [6], we have given theoretical proofs for the periodicity and occurrence times of the superrevivals.

One possibility for measuring the full and fractional superrevivals is to use a pump-probe method of detection for radial Rydberg wave packets with  $\bar{n} \approx 45 - 50$ . This is experimentally feasible, provided a delay line of 3 – 4 nsec is installed in the apparatus. This should permit detection of both full and fractional superrevivals.

We note that this analysis of wave-packet revivals is very general and can be applied to a variety of quantum systems other than Rydberg atoms [8]. In fact, a classification of different revival types is possible based solely on the form of  $E_n$  for quantum systems with discrete energy levels. For example, the energy levels of a particle in an infinite square well depend on  $n^2$ . Because of this quadratic dependence, the time scale  $t_{\text{sr}}$ , which depends on the third derivative of  $E_n$  with respect to  $n$ , is undefined. Wave packets in the infinite square well therefore exhibit perfect

full and fractional revivals and no superrevivals. A second example, in condensed matter physics, is of wave packets formed as superpositions of edge magnetoplasmons in quantum-Hall devices. We have shown that these wave packets can exhibit full and fractional revivals similar to those of Rydberg wave packets [9].

All of the results described above for hydrogen may be rederived in the context of a quantum defect theory based on atomic supersymmetry [7]. The wave functions of this theory form a complete and orthonormal set with the correct eigenenergies for an alkali-metal atom. They are particularly well suited for modeling the behavior of wave packets in alkali-metal atoms. The expansion of the energy for a Rydberg wave packet may be carried out in this context with the energies  $E_{n^*} = -1/2n^{*2}$ , where  $n^* = n - \delta(l)$  and  $\delta(l)$  is an asymptotic quantum defect for an alkali-metal atom. To investigate the effects of a laser detuning, the Taylor expansion is carried out around a noninteger central value  $N^*$  that may or may not be on resonance. For the off-resonance case, the noninteger part of  $N^*$  consists of two parts: one from the quantum defect and another from the laser detuning. In Ref. [5], we have shown that the effects of the quantum defects are different from those of the laser detuning. The time scales governing the evolution of the wave packet depend on the quantum defects differently from how they depend on the laser detuning.

### 3. Squeezed-State Description

The initial localization and classical behavior of radial Rydberg wave packets suggests that they might be described in terms of some kind of coherent state [16]. Indeed, at an early stage in the development of quantum mechanics, Schrödinger tried unsuccessfully to find nonspreading wave-packet solutions for a quantum-mechanical particle in a Coulomb potential evolving along a classical trajectory [17]. Many authors since have discussed this issue, and it is now known that there are no *exact* coherent states for the Coulomb problem [18, 19, 20, 21, 22, 23].

Although the original Schrödinger problem for the Coulomb potential has no solution, one can nonetheless obtain minimum-uncertainty wave packets exhibiting many features of the corresponding classical motion. For example, radial Rydberg wave packets follow the initial classical motion. However, they also exhibit distinctive quantum-mechanical features. In particular, their uncertainty product in  $r$  and  $p_r$  oscillates periodically as a function of time. This is a characteristic of a quantum-mechanical squeezed state [10].

To construct a squeezed-state description appropriate for radial Rydberg wave packets, we have adopted a procedure used in Refs. [20] in the context of the construction of generalized minimum-uncertainty coherent states. The idea is to change variables from  $r$  and  $p_r$  to a new set,  $R$  and  $P$ , chosen to have oscillatory dependence on a suitable variable. The similarities between the ensuing equations and the usual

quantum harmonic oscillator are sufficient to allow an analytical construction of our candidate Rydberg wave packets. Our method generates a three-parameter family of radial squeezed states

$$\psi(r) = \frac{(2\gamma_0)^{2\alpha+3}}{\Gamma(2\alpha+3)} r^\alpha e^{-\gamma_0 r} e^{-i\gamma_1 r} \quad . \quad (6)$$

To compare our radial squeezed states with other theory and experiment, we determine the parameters  $\alpha$ ,  $\gamma_0$ , and  $\gamma_1$  by fixing the form of the packet at the first pass through the classical apsidal point by the conditions  $\langle p_r \rangle = 0$ ,  $\langle r \rangle = r_{\text{out}}$ ,  $\langle H \rangle = E_{\bar{n}}$ , where  $r_{\text{out}}$  is the outer apsidal point of the orbit and  $E_{\bar{n}} = -1/2\bar{n}^2$ . We have shown that these radial squeezed states may be used as an initial wave function to model the motion of a wave packet produced by a short laser pulse [10]. The time evolution of the radial squeezed states exhibits the expected revival structure as well as the oscillations in the uncertainty product that are characteristic of a squeezed state.

We have also considered the problem of constructing squeezed states that are localized in three dimensions and which follow the classical trajectory of a particle moving on a keplerian ellipse. To construct a squeezed state localized in the angular coordinates requires identifying angular operators appropriate to a problem with spherical symmetry. These operators can then be used to obtain a class of squeezed states, called spherical squeezed states, which minimize uncertainty products for the angular variables. Combining the spherical squeezed states with radial squeezed states results in a class of minimum-uncertainty wave packets that are localized in all three dimensions and that travel along a keplerian ellipse. We call these three-dimensional wave packets keplerian squeezed states [11]. The widths of the wave packets oscillate during the motion, as is characteristic of squeezed states. The keplerian squeezed states maintain their shape for several cycles before collapsing and undergoing revivals.

Three-dimensional wave packets of this kind are of particular interest at present because experiments using short-pulsed lasers are attempting to produce Rydberg wave packets that move along elliptical orbits. To generate a three-dimensional wave packet localized in radial and angular coordinates, a superposition of  $n$ ,  $l$ , and  $m$  levels must be created. This requires the presence of additional fields during the excitation process. One proposal for achieving this involves using a short electric pulse to convert an angular state into a localized Rydberg wave packet moving on a circular orbit [24]. An additional weak electric field could then distort the orbit into an ellipse of arbitrary eccentricity. We expect the motion of these wave packets

to be well described by the keplerian squeezed states.

#### 4. Stark Wave Packets

The properties of Rydberg wave packets are being investigated in the presence of electric fields, which can significantly alter the atomic dynamics. Here, we address the question of whether fractional revivals can occur in these more complicated systems. Stark wave packets have energies that depend on two quantum numbers [13]. We prove below that under certain conditions Stark wave packets can exhibit full and fractional revivals. Moreover, we show the existence of new wave-packet behavior that does not occur for free wave packets [12].

A Stark wave packet is created when a short laser pulse excites a wave packet in the presence of a static external electric field. For a hydrogen atom in a weak electric field, the energies in atomic units are  $E_{nk} = -1/(2n^2) + 3nkF/2$ , where  $n$  is the principal quantum number,  $k = n_1 - n_2$  with  $n_1$  and  $n_2$  being parabolic quantum numbers, and  $F$  is the magnitude of the electric-field strength. We use a hydrogenic treatment to describe Stark wave packets. Wave packets for alkali-metal atoms can be studied using supersymmetry-based quantum defect theory. The time scales describing the evolution of hydrogenic and alkali-metal wave packets are not equal. However, we expect similar types of revival structure for both hydrogen and alkali-metal atoms.

We wish to examine the revival structure of a Stark wave packet  $\Psi(t)$  formed as a coherent superposition of states  $\phi_{nk}$  with energies  $E_{nk}$  depending on two quantum numbers  $n$  and  $k$ . We write  $\Psi(t) = \sum_{n,k} c_{nk} \phi_{nk} \exp[-iE_{nk}t]$ . The quantum number  $k$  is even or odd according to whether  $n$  is odd or even. Consider a superposition of Stark states centered around the values  $n = \bar{n}$  and  $k = \bar{k} = 0$ , and take the quantum number  $m$  associated with the third component of the angular momentum to be zero. The energy can then be expanded in a Taylor series around  $E_{\bar{n}\bar{k}}$ . We introduce the time scales

$$T_{\text{cl}}^{(n)} = \frac{2\pi}{\left(\frac{\partial E}{\partial n}\right)_{\bar{n},\bar{k}}} = 2\pi\bar{n}^3 \quad , \quad T_{\text{cl}}^{(k)} = \frac{2\pi}{2\left(\frac{\partial E}{\partial k}\right)_{\bar{n},\bar{k}}} = \frac{2\pi}{3F\bar{n}} \quad , \quad (7)$$

$$t_{\text{rev}}^{(n)} = \frac{2\pi}{\frac{1}{2}\left(\frac{\partial^2 E}{\partial n^2}\right)_{\bar{n},\bar{k}}} = \frac{4\pi}{3}\bar{n}^4 \quad , \quad t_{\text{rev}}^{(nk)} = \frac{2\pi}{2\left(\frac{\partial^2 E}{\partial n \partial k}\right)_{\bar{n},\bar{k}}} = \frac{2\pi}{3F} \quad . \quad (8)$$

No revival time  $t_{\text{rev}}^{(k)}$  is associated with the quantum number  $k$  since  $\partial^2 E / \partial k^2 = 0$ . Substituting these definitions into  $\Psi(t)$  and keeping terms to second order yields

$$\Psi(t) = \sum_{n,k} c_{nk} \phi_{nk} \exp \left[ -2\pi i \left( \frac{(n - \bar{n})t}{T_{\text{cl}}^{(n)}} + \frac{kt}{2T_{\text{cl}}^{(k)}} + \frac{(n - \bar{n})^2 t}{t_{\text{rev}}^{(n)}} + \frac{(n - \bar{n})kt}{2t_{\text{rev}}^{(nk)}} \right) \right] \quad . \quad (9)$$

For small times  $t$ , the first two terms of the time-dependent phase in Eq. (9) dominate. They represent beating between the two classical periods  $T_{\text{cl}}^{(n)}$  and  $T_{\text{cl}}^{(k)}$ . We say  $T_{\text{cl}}^{(n)}$  and  $T_{\text{cl}}^{(k)}$  are commensurate if  $T_{\text{cl}}^{(n)} = \frac{a}{b}T_{\text{cl}}^{(k)}$ , where  $a$  and  $b$  are relatively prime integers. If this relation holds, then the time evolution of  $\Psi(t)$  on short time scales exhibits a period  $T_{\text{cl}} = bT_{\text{cl}}^{(n)} = aT_{\text{cl}}^{(k)}$ .

For greater times, the revival time scales  $t_{\text{rev}}^{(n)}$  and  $t_{\text{rev}}^{(nk)}$  become relevant and modulate the initial behavior, causing the wave packet to spread and collapse. We find that the wave packet can undergo full revivals provided the revival times  $t_{\text{rev}}^{(n)}$  and  $t_{\text{rev}}^{(nk)}$  are commensurate and obey  $t_{\text{rev}}^{(n)} = (r/s)t_{\text{rev}}^{(nk)}$ , where  $r$  and  $s$  are relatively prime integers. The commensurability of the time scales depends on both  $\bar{n}$  and  $F$ . Restricting the electric-field strength  $F$  to less than the classical field-ionization threshold  $F_c = 1/16\bar{n}^4$  places limits on the ratios  $a/b$  and  $r/s$ . We find  $a/b < 3/16$  and  $r/s < 1/8$ . By tuning  $F$  to specific values, different commensurabilities and types of revival structure can be selected.

For fractional revivals to form in Stark wave packets, the wave function  $\Psi(t)$  in Eq. (9) must be expressible as a sum of distinct subsidiary wave functions. However, this can only occur at times  $t = t_{\text{frac}}$  that are simultaneously irreducible rational fractions of the two revival time scales. We define the times  $t_{\text{frac}} = (p_1/q_1)t_{\text{rev}}^{(n)} = (p_{12}/q_{12})t_{\text{rev}}^{(nk)}$ . Here, the pairs of integers  $(p_1, q_1)$  and  $(p_{12}, q_{12})$  are relatively prime. To prove that subsidiary wave packets form at the times  $t_{\text{frac}}$ , we first rewrite  $\Psi(t)$  in Eq. (9) by shifting  $(n - \bar{n}) \rightarrow n$  and separating the series into odd and even sums over  $n$ . We then let  $k \rightarrow 2k$  in the sum over odd  $n$ , and  $k \rightarrow 2k + 1$  in the sum over even  $n$ . This gives  $\Psi(t) = \Psi_{\text{odd}}(t) + \Psi_{\text{even}}(t)$ , where  $\Psi_{\text{odd}}(t)$  and  $\Psi_{\text{even}}(t)$  are the superpositions of odd- $n$  and even- $n$  states, respectively.

We define the doubly periodic wave functions  $\psi_{\text{cl}}^{(\text{odd})}$  and  $\psi_{\text{cl}}^{(\text{even})}$  as

$$\psi_{\text{cl}}^{(\text{odd/even})}(t_1, t_2) = \sum_{n \text{ odd/even}} \sum_k c_{nk} \phi_{nk} \exp \left( -2\pi i \left( \frac{nt_1}{T_{\text{cl}}^{(n)}} + \frac{kt_2}{T_{\text{cl}}^{(k)}} \right) \right) . \quad (10)$$

Next, consider the periodicity in  $n$  and  $k$  for the higher-order terms in the time-dependent phases of  $\Psi_{\text{odd}}(t)$  and  $\Psi_{\text{even}}(t)$  at  $t = t_{\text{frac}}$ . These terms are

$$\theta_{nk}^{(\text{odd})} = \frac{p_1}{q_1}n^2 - \frac{r}{s}\frac{p_1}{q_1}nk , \quad (11)$$

$$\theta_{nk}^{(\text{even})} = \frac{p_1}{q_1}n^2 - \frac{r}{s}\frac{p_1}{q_1}nk - \frac{r}{s}\frac{p_1}{q_1}\frac{1}{2}n . \quad (12)$$

Here,  $n$  is odd in Eq. (11) and even in Eq. (12). We seek the minimum periods  $l_1$ ,  $l_2$ ,  $l'_1$ , and  $l'_2$  such that  $\theta_{n+l_1, k}^{(\text{odd})} = \theta_{nk}^{(\text{odd})}$ ,  $\theta_{n, k+l_2}^{(\text{odd})} = \theta_{nk}^{(\text{odd})}$ ,  $\theta_{n+l'_1, k}^{(\text{even})} = \theta_{nk}^{(\text{even})}$ , and  $\theta_{n, k+l'_2}^{(\text{even})} = \theta_{nk}^{(\text{even})}$ . These relations yield four conditions for the periods  $l_1$ ,  $l_2$ ,  $l'_1$ , and  $l'_2$  in terms of  $n$ ,  $k$ , and  $t_{\text{frac}}$ .



Since the functions  $\psi_{\text{cl}}^{(\text{odd})}$  and  $\psi_{\text{cl}}^{(\text{even})}$  with  $t$  shifted by appropriate fractions of  $T_{\text{cl}}^{(n)}$  and  $T_{\text{cl}}^{(k)}$  have the same periodicities in  $n$  and  $k$  as  $\theta_{nk}^{(\text{odd})}$  and  $\theta_{nk}^{(\text{even})}$ , respectively, we may use these functions as a basis for an expansion of the wave functions  $\Psi_{\text{odd}}(t)$  and  $\Psi_{\text{even}}(t)$  at the times  $t_{\text{frac}}$ . The result is

$$\begin{aligned} \Psi(t) = & \sum_{s_1=0}^{l_1-1} \sum_{s_2=0}^{l_2-1} a_{s_1 s_2}^{(\text{odd})} \psi_{\text{cl}}^{(\text{odd})} \left( t + \frac{s_1}{l_1} T_{\text{cl}}^{(n)}, t + \frac{s_2}{l_2} T_{\text{cl}}^{(k)} \right) \\ & + e^{-i\pi \left( \frac{p_{12} t_{\text{rev}}^{(nk)}}{q_{12} T_{\text{cl}}^{(k)}} \right)} \sum_{s_1=0}^{l'_1-1} \sum_{s_2=0}^{l'_2-1} a_{s_1 s_2}^{(\text{even})} \psi_{\text{cl}}^{(\text{even})} \left( t + \frac{s_1}{l'_1} T_{\text{cl}}^{(n)}, t + \frac{s_2}{l'_2} T_{\text{cl}}^{(k)} \right) . \end{aligned} \quad (13)$$

The expansion coefficients  $a_{s_1 s_2}^{(\text{odd})}$  and  $a_{s_1 s_2}^{(\text{even})}$  are

$$a_{s_1 s_2}^{(\text{odd})} = \frac{1}{l_1 l_2} \sum_{\kappa_1=0}^{l_1-1} \sum_{\kappa_2=0}^{l_2-1} \exp \left( 2\pi i \theta_{\kappa_1 \kappa_2}^{(\text{odd})} \right) \exp \left( 2\pi i \frac{s_1}{l_1} \kappa_1 \right) \exp \left( 2\pi i \frac{s_2}{l_2} \kappa_2 \right) , \quad (14)$$

$$a_{s_1 s_2}^{(\text{even})} = \frac{1}{l'_1 l'_2} \sum_{\kappa_1=0}^{l'_1-1} \sum_{\kappa_2=0}^{l'_2-1} \exp \left( 2\pi i \theta_{\kappa_1 \kappa_2}^{(\text{even})} \right) \exp \left( 2\pi i \frac{s_1}{l'_1} \kappa_1 \right) \exp \left( 2\pi i \frac{s_2}{l'_2} \kappa_2 \right) , \quad (15)$$

When these expressions are substituted into Eq. (13) and the definitions (10) are used, Eq. (13) reduces to the form given in Eq. (9). This completes the proof of the formation of fractional revivals for Stark wave packets.

As an illustrative example, consider the case  $\bar{n} = 24$ , and set  $t_{\text{rev}}^{(n)}/t_{\text{rev}}^{(nk)} = r/s = 1/12$  by tuning the electric-field strength to  $F \simeq 645.8$  volts/cm. For this example,  $t_{\text{rev}} = t_{\text{rev}}^{(nk)} = 12t_{\text{rev}}^{(n)}$ . We find at  $t \approx t_{\text{rev}}$ ,

$$\Psi(t) \approx \psi_{\text{cl}}^{(\text{odd})}(t, t) + \psi_{\text{cl}}^{(\text{even})}(t, t) . \quad (16)$$

These sums are in phase and combine as a single total wave packet, producing the full revival at  $t_{\text{rev}}$ .

At  $t = t_{\text{rev}}/2$ , however, we find a time phase between  $\Psi_{\text{odd}}(t)$  and  $\Psi_{\text{even}}(t)$ , with the full wave function reducing to

$$\Psi(t) \approx \psi_{\text{cl}}^{(\text{odd})} \left( t, t + \frac{1}{2} T_{\text{cl}}^{(k)} \right) + \psi_{\text{cl}}^{(\text{even})} \left( t + \frac{1}{4} T_{\text{cl}}^{(n)}, t \right) . \quad (17)$$

We see that this fractional revival consists of two subsidiary wave functions out of phase with each other.

Due to the unique behavior of the quantum number  $k$ , the functions  $\psi_{\text{cl}}^{\text{odd}}$  and  $\psi_{\text{cl}}^{\text{even}}$  exhibit additional time behavior depending on  $\frac{1}{2}T_{\text{cl}}^{(n)}$ . The functions  $\psi_{\text{cl}}^{\text{odd}}$  are antiperiodic with period  $\frac{1}{2}T_{\text{cl}}^{(n)}$ , while  $\psi_{\text{cl}}^{\text{even}}$  is periodic in  $\frac{1}{2}T_{\text{cl}}^{(n)}$ . They obey the relations,

$$\psi_{\text{cl}}^{(\text{odd})} \left( t + \frac{1}{2} T_{\text{cl}}^{(n)}, t \right) = -\psi_{\text{cl}}^{(\text{odd})}(t, t) , \quad (18)$$

$$\psi_{\text{cl}}^{(\text{even})}(t + \frac{1}{2}T_{\text{cl}}^{(n)}, t) = \psi_{\text{cl}}^{(\text{even})}(t, t) \quad . \quad (19)$$

It is this additional behavior with period  $\frac{1}{2}T_{\text{cl}}^{(n)}$  that causes Stark wave packets to have unconventional revival structure. In Ref. [12], we proved that at the  $t = t_{\text{rev}}/2$  fractional revival, the antiperiodic behavior of  $\psi_{\text{cl}}^{(\text{odd})}$  causes nodes to appear in the autocorrelation function with a periodicity equal to  $\frac{1}{2}T_{\text{cl}}^{(n)}$ . However, no additional nodes or periodicities appear for the even states. This additional interference behavior has no analogue for the case of free Rydberg wave packets.

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